Berry phase in an entangled spin-$\frac{1}{2}$ system

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The influence of the geometric phase, in particular the Berry phase, on an entangled spin-$\frac{1}{2}$ system is studied. The case, where the geometric phase is generated only by one part of the Hilbert space is discussed. The effects of the dynamical phase can be cancelled by using the "spin-echo" method. The analysis how the Berry phase affects the violation of a Bell inequality is presented. Furthermore an experimental realization of the setup within neutron interferometry is proposed.

1 Introduction

Since the work of John S. Bell [1] entanglement of quantum states became an important development in physics. Entanglement is closely related to the violation of Bell inequalities (BI) which distinguish quantum mechanics (QM) from (all) local realistic theories [2]. Experiments with photons in this field (see, e.g., Ref. [3] for a review) confirm QM with its nonlocal feature in an impressive way. In the last years there have also been considerable activities to test entangled massive systems in particle physics [4,5]. Since entanglement is the basis for quantum communication and quantum information (see Ref. [3]) it became an important issue of investigation nowadays.

Geometric phases such as the Berry phase [6] play a considerable role in physics and arise in a quantum system when its time evolution is cyclically and adiabatically. Its deep geometric origin is given by a holonomy of the line bundle of the states where the phase emerges from the integral of the connection (or curvature) of the bundle over the parameter space [7]. Experimentally, geometric phases have been tested in various cases, e.g., with photons [8,9] and neutrons [10,11].

Whereas the geometric phase in a single particle system is already studied very well, both theoretically and experimentally, its effect on entangled quantum systems is less known. However, there is increasing interest to combine both quantum phenomena, the geometric phase and the entanglement of a system [12,13].

In the following we present the idea of Berry to construct a geometric phase which is discussed in detail for the spin-$\frac{1}{2}$ case, where the Berry phase is implemented by an adiabatic rotating magnetic field. The influence of the

Berry phase on the entanglement of a spin-$\frac{1}{2}$ system by considering a BI is shown and an explicit experimental setup within neutron interferometry [14,15] is proposed. In particular, when using a polarized beam we have entanglement between different degrees of freedom, i.e., the spin and the path of the neutron. In this case it is physically rather noncontextuality than locality which is tested experimentally [16].

2 The Berry phase

The Berry phase [6] arises in systems which are embedded in an environment described by a time-dependent (possibly multidimensional) parameter $R(t)$. Therefore the time evolution of the system is given by the following Schrödinger equation

$$H(R(t))|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle \;, \tag{1}$$

where the Hamiltonian $H(R(t))$ depends on the external parameter.

Two conditions have to be fulfilled in order to get a Berry phase. First we take into account only cyclic evolutions of the system, where $R(T) = R(0)$ and also $H(R(T)) = H(R(0))$. Furthermore we demand an adiabatic (or slow) time evolution [17], which means that the external time τ of the evolution is much slower than an internal timescale defined by $t_i = h/(E_i - E_j)$: $\tau \gg t_i$. The eigenvalues of the system should not be degenerated and there should occur no level-crossing of the energy eigenvalues during the evolution.

Suppose we start the evolution of the system in the n-th eigenstate of the Hamiltonian $H(R)$,

$$|\psi(0)\rangle = |n(R(0))\rangle \;, \tag{2}$$

where for any time t the relation $H(R(t)) = E_n|n(R(t))\rangle\langle n(R(t))|$ is valid. It turns out that the final state of the cyclic evolution can be written in the following way

$$|\psi(\tau)\rangle = e^{i\theta_n} e^{i\gamma_n}|n(R(0))\rangle \;, \tag{3}$$

where θ_n denotes the usual dynamical phase given by

$$\theta_n(\tau) = -\frac{1}{\hbar}\int_0^\tau E_n(t)dt \;, \tag{4}$$

and γ_n, the additional geometric or Berry phase, is derived to

$$\gamma_n(C) = i \oint_C \langle n(R)|\nabla_R|n(R)\rangle dR \;. \tag{5}$$

The geometric phase is independent of the parameterization of the curve C, it only depends on the geometry of the parameter space $\{R\}$.

3 Example: the Berry phase for a spin-$\frac{1}{2}$ particle

Now we concentrate on the spin-$\frac{1}{2}$ system where it is rather simple to construct explicitly a scenario where a geometric phase arises.

The scenario is as follows. The particle, without loss of generality moving in y-direction, couples to a time dependent magnetic field

$$\vec{B}(t) = B \begin{pmatrix} \sin\vartheta\cos(\omega_0 t) \\ \sin\vartheta\sin(\omega_0 t) \\ \cos\vartheta \end{pmatrix} = B \cdot \vec{n}(\vartheta; t) \;, \tag{6}$$

with unit vector $\vec{n}(\vartheta; t)$ and constant norm $B \equiv |\vec{B}(t)|$, which rotates adiabatically with an angle velocity of ω_0 around the z-axes under an angle ϑ. The interaction is described by the Hamiltonian

$$H(t) = \frac{\mu}{2}\vec{B}\cdot\vec{\sigma} = \frac{\mu}{2}B \begin{pmatrix} \cos\vartheta & e^{-i\omega t}\sin\vartheta \\ e^{i\omega t}\sin\vartheta & -\cos\vartheta \end{pmatrix} \;, \tag{7}$$

where the coupling constant is given by $\mu = g\mu_B$, the Landé factor g times the Bohr magneton $\mu_B = \frac{1}{2}\frac{e}{m}\hbar$.

The instantaneous eigenstates of the spin-operator in direction $\vec{n}(\vartheta; t)$ expanded in the σ_z-basis are given by

$$\begin{aligned} |\Uparrow_n; t\rangle &= \cos\frac{\vartheta}{2}\; |\Uparrow_z\rangle + \sin\frac{\vartheta}{2}\; e^{i\omega_0 t}\; |\Downarrow_z\rangle \\ |\Downarrow_n; t\rangle &= -\sin\frac{\vartheta}{2}\; |\Uparrow_z\rangle + \cos\frac{\vartheta}{2}\; e^{i\omega_0 t}\; |\Downarrow_z\rangle \;, \end{aligned} \tag{8}$$

and the corresponding time independent energy levels are

$$E_\pm = \pm\frac{\mu B}{2} = \pm\omega_1\hbar \;, \tag{9}$$

where $\omega_1 := \frac{E_+ - E_-}{2\hbar} = \frac{\mu B}{2\hbar}$ denotes the energy difference of spin $\Uparrow_n$ and spin $\Downarrow_n$ and represents the characteristic frequency of the system.

Let us consider an adiabatic (which means $\frac{\omega_0}{\omega_1} \ll 1$) and cyclic time evolution for the period $\tau = \frac{2\pi}{\omega_0}$ of these eigenstates. Then each eigenstate picks up a phase factor that can be split into a geometrical and dynamical part of the following form

$$\begin{aligned} |\Uparrow_n; t=0\rangle &\longrightarrow |\Uparrow_n; t=\tau\rangle = e^{i\gamma_+(\vartheta)} e^{i\theta_+} |\Uparrow_n; t=0\rangle \\ |\Downarrow_n; t=0\rangle &\longrightarrow |\Downarrow_n; t=\tau\rangle = e^{i\gamma_-(\vartheta)} e^{i\theta_-} |\Downarrow_n; t=0\rangle \;, \end{aligned} \tag{10}$$

with

$$\gamma_+(\vartheta) = -\pi(1-\cos\vartheta) \qquad \gamma_-(\vartheta) = -\pi(1+\cos\vartheta) = -\gamma_+(\vartheta) - 2\pi \tag{11}$$

$$\theta_+ = -\frac{1}{\hbar}E_+\tau = -2\pi\frac{\omega_1}{\omega_0} \qquad \theta_- = +\frac{1}{\hbar}E_-\tau = +2\pi\frac{\omega_1}{\omega_0} = -\theta_+ \ , \tag{12}$$

where $\gamma_\pm$ denotes the Berry phase which is precisely half of the solid angle $\frac{1}{2}\Omega$ swept out by the magnetic field during the evolution and $\theta_\pm$ is the dynamical phase.

4 The spin – echo method

The dynamical effects which would dominate the geometrical one can be eliminated by using the so called "spin-echo" method. First the propagating particle is subjected to the rotating magnetic field in the direction $\vec{n}(\vartheta)$ for one period and therefore picks up the phases given by Eq.(10). Afterwards the particle passes another rotating field which points in direction $-\vec{n}(\pi-\vartheta)$ again for one period. Then the states change according to

$$\begin{aligned} |\Uparrow_n\rangle \equiv |\Downarrow_{-n}\rangle &\longrightarrow e^{i\gamma_-(\pi-\vartheta)}e^{i\theta_-}|\Downarrow_{-n}\rangle \equiv e^{i\gamma_+(\vartheta)}e^{i\theta_-}|\Uparrow_n\rangle \\ |\Downarrow_n\rangle \equiv |\Uparrow_{-n}\rangle &\longrightarrow e^{i\gamma_+(\pi-\vartheta)}e^{i\theta_+}|\Uparrow_{-n}\rangle \equiv e^{i\gamma_-(\vartheta)}e^{i\theta_+}|\Downarrow_n\rangle \ , \end{aligned} \tag{13}$$

where the geometrical phases can be determined according to

$$\gamma_\pm(\pi-\vartheta) = -\pi(1\mp\cos(\pi-\vartheta)) = -\pi(1\pm\cos\vartheta) = \gamma_\mp(\vartheta) \ . \tag{14}$$

Therefore we get the following net-effect after two rotation-periods

$$|\Uparrow_n\rangle \to e^{2i\gamma_+}|\Uparrow_n\rangle \qquad |\Downarrow_n\rangle \to e^{2i\gamma_-}|\Downarrow_n\rangle \ , \tag{15}$$

or if we use two half-periods of rotation we get

$$|\Uparrow_n\rangle \to e^{i\gamma_+}|\Uparrow_n\rangle \qquad |\Downarrow_n\rangle \to e^{i\gamma_-}|\Downarrow_n\rangle \ , \tag{16}$$

where the dynamical effects totally disappear.

5 The Berry phase and the entangled state

Let us consider an entangled state of two spin-$\frac{1}{2}$ particles, e.g., the antisymmetric Bell singlet state $\Psi^{(-)}$. One of the particles (e.g., the left side moving particle) interacts twice with the adiabatically rotating magnetic fields as described in Sect.4. Thus only one subspace of the Hilbert space is influenced by the phases, see Fig.1.

To locate the Berry phase we decompose the initial Bell singlet state into the eigenstates of the interaction Hamiltonian

$$|\Psi(t=0)\rangle = |\Psi^{(-)}\rangle = \frac{1}{\sqrt{2}}\big\{|\Uparrow_n\Downarrow_n\rangle - |\Downarrow_n\Uparrow_n\rangle\big\}\ . \qquad (17)$$

According to our "spin-echo" construction, after one cycle, this state picks up precisely the geometric phase (16)

$$|\Psi(t=\tau)\rangle = \frac{1}{\sqrt{2}}\big\{e^{i\gamma_+}|\Uparrow_n\Downarrow_n\rangle - e^{i\gamma_-}|\Downarrow_n\Uparrow_n\rangle\big\}\ , \qquad (18)$$

which can be rewritten by neglecting an overall phase factor

$$|\Psi(t=\tau)\rangle = \frac{1}{\sqrt{2}}\big\{|\Uparrow_n\Downarrow_n\rangle - e^{-2i\gamma_+}|\Downarrow_n\Uparrow_n\rangle\big\}\ . \qquad (19)$$

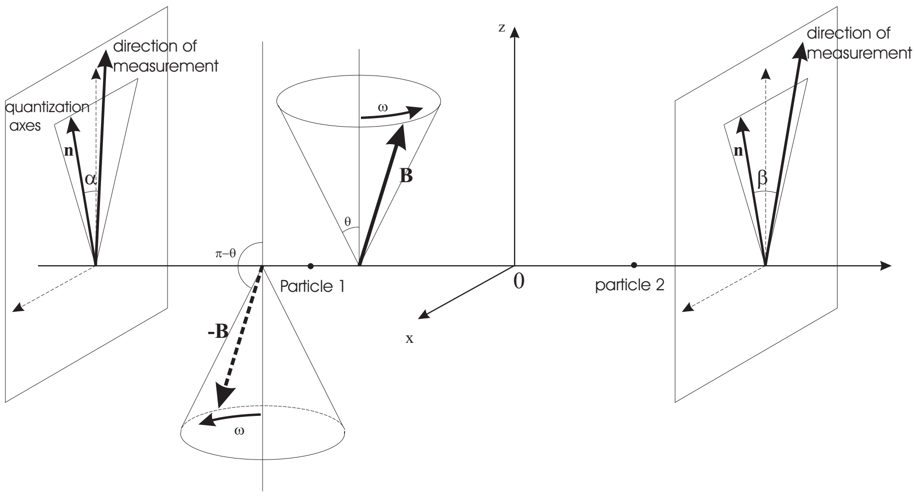

Figure 1: Schematic view of the setup.

Now we perform a common Bell experiment. Therefore we have to consider the joint probability of finding spin-up on the left side under an angle α from the quantization axis $\vec{n}$ and spin-up (spin-down) on the right side under an angle β, see Fig.1, which are derived to

$$
\begin{aligned}
&P(\alpha \Uparrow_n, \beta \Uparrow_n) \\
&= \Big\| \Big(\langle \Uparrow_n | \cos\frac{\alpha}{2} + \langle \Downarrow_n | \sin\frac{\alpha}{2} \Big) \otimes \Big(\langle \Uparrow_n | \cos\frac{\beta}{2} + \langle \Downarrow_n | \sin\frac{\beta}{2} \Big) | \Psi(t=\tau) \rangle \Big\|^2 \\
&= \frac{1}{4}\big(1 - \cos\alpha\cos\beta - \cos(2\gamma_+)\sin\alpha\sin\beta\big) \;,
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
&P(\alpha \Uparrow_n, \beta \Downarrow_n) \\
&= \Big\| \Big(\langle \Uparrow_n | \cos\frac{\alpha}{2} + \langle \Downarrow_n | \sin\frac{\alpha}{2} \Big) \otimes \Big(-\langle \Uparrow_n | \sin\frac{\beta}{2} + \langle \Downarrow_n | \cos\frac{\beta}{2} \Big) | \Psi(t=\tau) \rangle \Big\|^2 \\
&= \frac{1}{4}\big(1 + \cos\alpha\cos\beta + \cos(2\gamma_+)\sin\alpha\sin\beta\big) \;.
\end{aligned} \tag{21}
$$

The expectation value of the joint measurement is given by

$$
\begin{aligned}
E(\alpha,\beta) &= 2(P(\alpha \Uparrow_n, \beta \Uparrow_n) - P(\alpha \Uparrow_n, \beta \Downarrow_n)) \\
&= -\cos\alpha\cos\beta - \cos(2\gamma_+)\;\sin\alpha\sin\beta \;.
\end{aligned} \tag{22}
$$

To test the local features of the system we consider the CHSH-inequality (Clauser, Horne, Shimony, Holt) [19], which is an experimentally testable type of a Bell inequality

$$
S \leq 2 \;, \tag{23}
$$

where the S-function is given by

$$
\begin{aligned}
S(\alpha,\beta,\gamma,\delta;\gamma_+) &= \big| E(\alpha,\beta) - E(\alpha,\gamma) + E(\delta,\beta) + E(\delta,\gamma) \big| \\
&= \Big| -\cos\alpha\cos\beta + \cos\alpha\cos\gamma - \cos\delta\cos\beta - \cos\delta\cos\gamma \\
&\quad - \cos(2\gamma_+)\big(\sin\alpha\sin\beta - \sin\alpha\sin\gamma + \sin\delta\sin\beta + \sin\delta\sin\gamma\big) \Big| \;.
\end{aligned} \tag{24}
$$

The original inequality without Berry phases is maximally violated by the value $S_{\max} = 2\sqrt{2}$ for the Bell angles $\alpha_B = 0$, $\beta_B = \frac{\pi}{4}$, $\gamma_B = \frac{3\pi}{4}$, $\delta_B = \frac{\pi}{2}$. If a Berry phase is present in the system the S-function depends on the geometric phase and for fixed Bell angles we get

$$
S(\alpha_B,\beta_B,\gamma_B,\delta_B;\gamma_+) = \sqrt{2}(1 + \cos(2\gamma_+)) \;, \tag{25}
$$

which is plotted in Fig.2 (For recent developments and a more detailed analysis of the S-function see Ref. [20].)

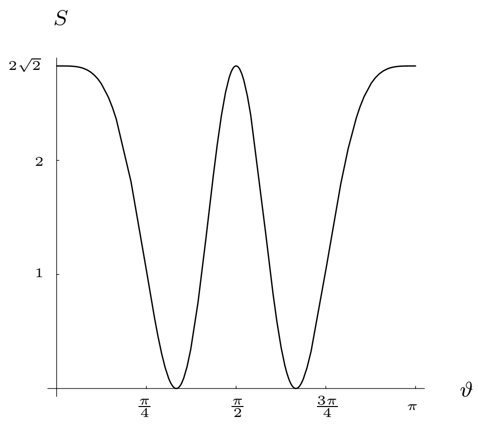

Figure 2: The S-function, Eq.(25), for the Berry phase on one subspace.

6 Proposed neutron-experiment

Neutron interferometry is an almost ideal tool to investigate spin-$\frac{1}{2}$ systems. Furthermore in a recent experiment of Hasegawa et al. [16] it was possible to create entanglement in the neutron system where different degrees of freedom (path and spin) got entangled. In this case Bell inequalities test noncontextuality rather than locality.

In our polarized neutron interferometer experiment the wave function of each neutron is defined over a tensor product of Hilbert spaces which describe the spatial and spin components of the wave function and is entangled analogously to the two spin-$\frac{1}{2}$ system

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big\{|\mathrm{I}\rangle \otimes |\Uparrow_n\rangle - |\mathrm{II}\rangle \otimes |\Downarrow_n\rangle\big\}\ . \tag{26}$$

The states $|\Uparrow_n\rangle$ and $|\Downarrow_n\rangle$ denote the spin states of beam-I and -II, as well as $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$ the states in the two beam paths in the interferometer.

A schematic view of the experimental setup is shown in Fig.3. In addition to an auxiliary phase shifter, two radio-frequency (RF) spin-flippers are inserted into one beam path and a direct current (DC) π-spin-flipper into the other path of the interferometer. The former two flippers enable the neutron spinors to evolve along a particular curve inducing only a geometric phase γ_B without any dynamical component, see Fig.4. The latter flipper produces the entangled state, like $|\Psi(t=\tau)\rangle$ in Eq.(19). Thus, after the spinor evolution the total wave function is represented by

$$|\Psi(\gamma_B)\rangle = \frac{1}{\sqrt{2}}\Big\{|\mathrm{I}\rangle \otimes |\Uparrow_n\rangle - \mathrm{e}^{i\gamma_\mathrm{B}}\,|\mathrm{II}\rangle \otimes |\Downarrow_\mathrm{n}\rangle\Big\}\ , \tag{27}$$

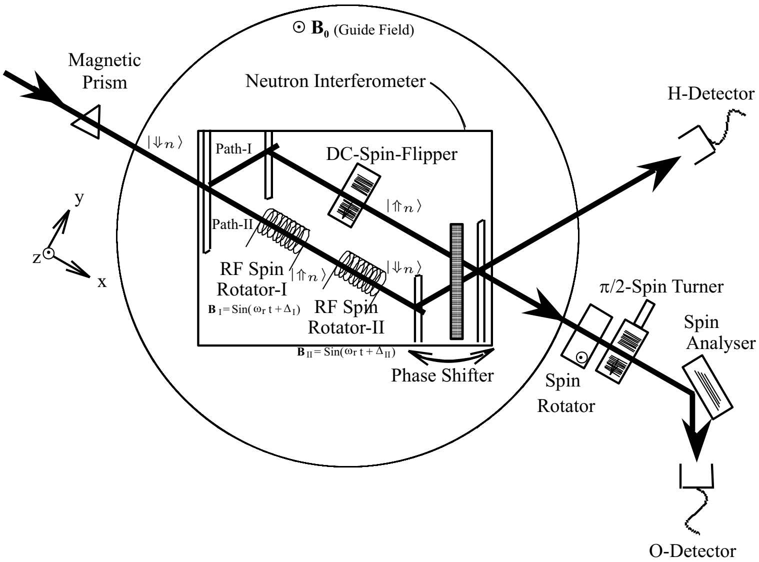

Figure 3: Schematic view of the experimental setup.

with the geometrical phase

$$\gamma_B = \frac{1}{2}\Omega = \phi_1 - \phi_2 \ . \tag{28}$$

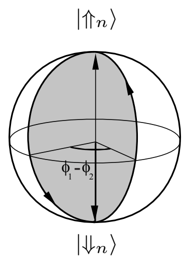

Figure 4: Schematic representation of the spinor evolution with the use of the Poincaré sphere.

Joint measurements on the path and spin are performed by choosing the phase shift and the angle of the spinor analysis appropriately. The measurements are expected to recover the predicted behavior of the S-function. The experiment is in progress.

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